

# MTH250 - CALCULUS

## Lecture No. 22

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

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# LECTURE 22

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## Vector valued functions

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### 22.1 Scalar triple product

If

$\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ ,  $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$  and  $\vec{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$  are vectors then the number  $\vec{u} \cdot (\vec{v} \times \vec{w})$  is called the scalar triple product of  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$ . The value of scalar triple product can be obtained directly from the formula

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

**Theorem 22.1.** *Let  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  be nonzero vectors. The volume  $V$  of the parallelepiped that has  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  as adjacent edges is*

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|.$$

*Moreover,  $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$  if and only if  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  lies in the same plane (coplaner).*

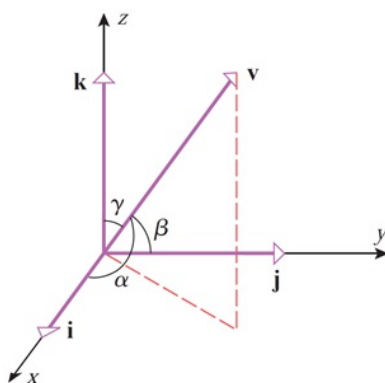


Figure 22.1: Direction cosines.

## 22.2 Direction cosines

**Definition 22.2** (Direction cosines).

The direction cosines of a nonzero vector  $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$  are defined by

$$\cos \alpha = \frac{v_1}{|\vec{v}|}, \quad \cos \beta = \frac{v_2}{|\vec{v}|}, \quad \cos \gamma = \frac{v_3}{|\vec{v}|}.$$

By definition

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

**Example 22.1.** Find the direction cosines of vector  $\vec{v} = 2\hat{i} - 4\hat{j} + 4\hat{k}$ .

*Solution.* As  $|\vec{v}| = \sqrt{4 + 16 + 16} = 6$ , thus

$$\frac{\vec{v}}{|\vec{v}|} = \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}.$$

Consequently, we have

$$\cos \alpha = \frac{1}{3}, \quad \cos \beta = -\frac{2}{3}, \quad \cos \gamma = \frac{2}{3}.$$

□

## 22.3 Parametric equations of lines

**Definition 22.3** (Parametric equation).

Suppose that a particle moves along a curve  $C$  in the  $xy$ -plane in such a way that its  $x$ - and  $y$ -coordinates, as functions of a parameter  $t$ , are

$$x = f(t), \quad y = g(t).$$

We call these the parametric equations of motion for the particle and refer to  $C$  as the trajectory of the particle or the graph of the equations.

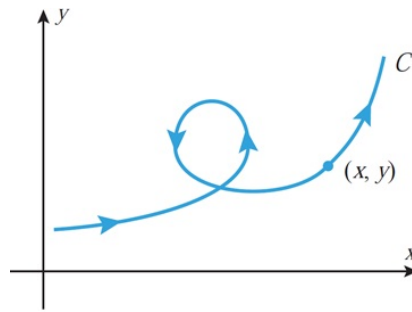
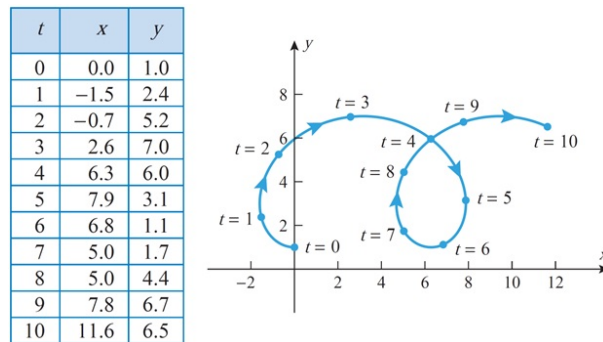


Figure 22.2: Parametric curve.

**Example 22.2.** Consider the particle whose parametric equations of motions are

$$x = t - 3 \sin t, \quad y = 4 - 3 \cos t$$

Its trajectory over the time interval  $0 \leq t \leq 10$  is sketched in the Figure 22.3.

Figure 22.3: Parametric curve defined by  $x = t - 3 \sin t$ ,  $y = 4 - 3 \cos t$ .

**Definition 22.4** (Parametric equation of lines).

A line in three dimension that passes through the point  $P_0(x_0, y_0, z_0)$  and is parallel to the nonzero vector

$$\vec{v} = (a, b, c) = a\hat{i} + b\hat{j} + c\hat{k}$$

has parametric equations

$$\begin{cases} x = x_0 + at, \\ y = y_0 + bt, \\ z = z_0 + ct. \end{cases}$$

**Example 22.3.** Find the parametric equations of the line passing through  $(1, 2, -3)$  and parallel to  $\vec{v} = 4\hat{i} + 5\hat{j} - 7\hat{k}$ .

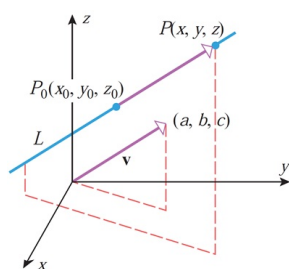


Figure 22.4: Line passing through  $P_0(x_0, y_0, z_0)$  and parallel to  $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ .

*Solution.* The line has parametric equations

$$x = 1 + 4t, \quad y = 2 + 5t, \quad z = -3 - 7t.$$

□

**Example 22.4.** Find the parametric equations of the line passing through  $P_1(2, 4, -1)$  and  $P_2(5, 0, 7)$ .

*Solution.* The vector  $\text{vec}P_1P_2 = (3, -4, 8)$  is parallel to the required and the point  $P_1(2, 4, -1)$  lies on the line, so it follows that the line has parametric equations

$$x = 2 + 3t, \quad y = 4 - 4t, \quad z = -1 + 8t.$$

□

## 22.4 Vector equations of lines

As two vectors are equal if and only if their components are equal. So the parametric equations of line can be rewritten in the form

$$(x, y, z) = (x_0 + ta, y_0 + tb, z_0 + tc),$$

or equivalently, as

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c).$$

If we define

$$\vec{r} = (x, y, z), \quad \vec{r}_0 = (x_0, y_0, z_0), \quad \vec{v} = (a, b, c).$$

Then:

**Definition 22.5** (Vector equation of lines). The vector equation of line is

$$\vec{r} = \vec{r}_0 + t\vec{v}.$$

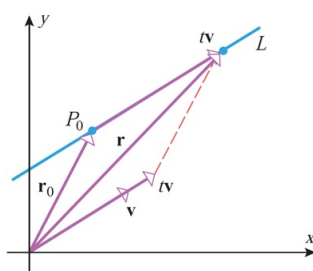


Figure 22.5: Vector equation of line.

**Example 22.5.** Find parametric equations of the line  $L$  passing through the points  $P_1(2, 4, -1)$  and  $P_2(5, 0, 7)$ .

*Solution.* The vector  $\vec{P_1P_2} = (3, -4, 8)$  is parallel to the line, so it can be used as  $\vec{v}$ . For  $\vec{r}_0$  we can use either the vector from the origin to  $P_1$  or the vector from the origin to  $P_2$ . Using the former yields

$$\vec{r}_0 = (2, 4, -1).$$

Thus, a vector equation of the line through  $P_1$  and  $P_2$  is

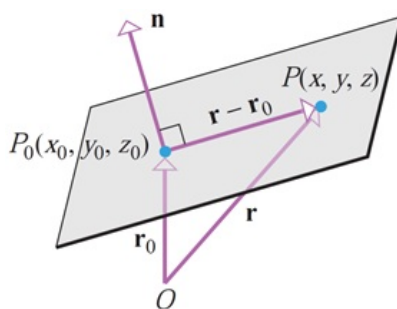
$$(x, y, z) = (2, 4, -1) + t(3, -4, 8).$$

□

## 22.5 Vector equations of planes

**Definition 22.6** (Normal vector).

A vector perpendicular to a plane is called a normal to the plane.

Figure 22.6: Plane with normal  $\vec{n}$ , and containing points  $P_0(x_0, y_0, z_0)$  and  $P(x, y, z)$ .

Suppose, we want to find the equation of the plane passing through  $P_0(x_0, y_0, z_0)$  and perpendicular to the vector  $\mathbf{n} = (a, b, c)$ . Define the vectors

$$\vec{r} = (x, y, z) \quad \text{and} \quad \vec{r}_0 = (x_0, y_0, z_0).$$

The plane consists precisely of those points  $P(x, y, z)$  for which  $\vec{r} - \vec{r}_0$  is orthogonal to  $\mathbf{n}$  i.e.

$$\mathbf{n} \cdot (\vec{r} - \vec{r}_0) = 0.$$

We can also express this vector equation of a plane in terms of components as

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

from which we obtain

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This is called the *point-normal* form of the equation of a plane.

**Theorem 22.7.** *If  $a, b, c \neq 0$  and  $k$  are constants, then the graph of the equation*

$$ax + by + cz + d = 0$$

*is a plane that has the vector  $\mathbf{n} = (a, b, c)$  as a normal.*

**Example 22.6.** *Find an equation of the plane through the points  $P_1(1, 2, -1)$ ,  $P_2(2, 3, 1)$  and  $P_3(3, -1, 2)$ .*

*Solution.* Since the points  $P_1$ ,  $P_2$ , and  $P_3$  lie in the plane, the vectors  $\vec{P_1P_2} = (1, 1, 2)$  and  $\vec{P_1P_3} = (2, -3, 3)$  are parallel to the plane. Therefore,

$$\vec{P_1P_2} \times \vec{P_1P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{vmatrix} = 9\hat{i} + \hat{j} - 5\hat{k}$$

is normal to the plane, since it is orthogonal to both  $\vec{P_1P_2}$  and  $\vec{P_1P_3}$ . By using this normal and the point  $P_1(1, 2, -1)$  in the plane, we obtain the point-normal form

$$9(x - 1) + (y - 2) - 5(z + 1) = 0$$

which can be rewritten as

$$9x + y - 5z - 16 = 0.$$

□

**Example 22.7.** *Determine whether the line*

$$x = 3 + 8t, \quad y = 4 + 5t, \quad z = -3 - t$$

*is parallel to the plane  $x - 3y + 5z = 12$ .*

*Solution.* The vector  $\mathbf{v} = (8, 5, -1)$  is parallel to the line and the vector  $\mathbf{n} = (1, -3, 5)$  is normal to the plane. For the line and plane to be parallel, the vectors  $\vec{v}$  and  $\mathbf{n}$  must be orthogonal. But this is not so, since the dot product

$$\vec{v} \cdot \mathbf{n} = (8)(1) + (5)(-3) + (-1)(5) = -12$$

is nonzero. Thus, the line and plane are not parallel.  $\square$

## 22.6 Vector-valued functions

**Definition 22.8** (Vector-valued function). A vector-valued function, or vector function, is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

For every number  $t$  in the domain of  $\mathbf{r}$  there is a unique vector denoted by  $\mathbf{r}(t) \in \mathbb{R}^3$ .

If  $f(t)$ ,  $g(t)$ , and  $h(t)$  are the components of the vector  $\mathbf{r}(t)$ , then  $f$ ,  $g$ , and  $h$  are real-valued functions called the component functions of  $\mathbf{r}$  and we can write

$$\mathbf{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}.$$

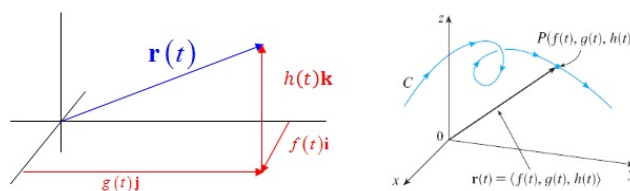


Figure 22.7: Vector valued function  $\mathbf{r}$  and its component functions.

**Definition 22.9.** The domain of a vector-valued function  $\mathbf{r}(t)$  is the set of allowable values for  $t$ . If  $\mathbf{r}(t)$  is defined in terms of component functions and the domain is not specified explicitly, then it will be understood that the domain is the intersection of the natural domains of component functions; this is called the natural domain of  $\mathbf{r}(t)$ .

**Example 22.8.** Find the natural domain of

$$\mathbf{r}(t) = (\ln|t-1|, e^t, \sqrt{t}) = \ln|t-1|\hat{i} + e^t\hat{j} + \sqrt{t}\hat{k}.$$

*Solution.* The natural domains of the component functions

$$x(t) = \ln|t-1|, \quad y(t) = e^t, \quad z(t) = \sqrt{t}$$

are

$$(-\infty, 1) \cup (1, +\infty), \quad (-\infty, +\infty), \quad [0, +\infty)$$

respectively. The intersection of these sets is

$$[0, 1) \cup (1, +\infty)$$

so the natural domain of  $\mathbf{r}(t)$  consists of all values of  $t$  such that

$$0 \leq t < 1 \quad \text{or} \quad t > 1.$$

□

## 22.7 Limits, continuity & differentiability

**Definition 22.10** (Limits of vector-valued functions).

Let  $\mathbf{r}(t)$  be a vector-valued function that is defined for all  $t$  in some open interval containing the number  $a$ , except that  $\mathbf{r}(t)$  need not be defined at  $a$ . We will write

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L} \quad \text{if and only if} \quad \lim_{t \rightarrow a} |\mathbf{r}(t) - \mathbf{L}| = 0.$$

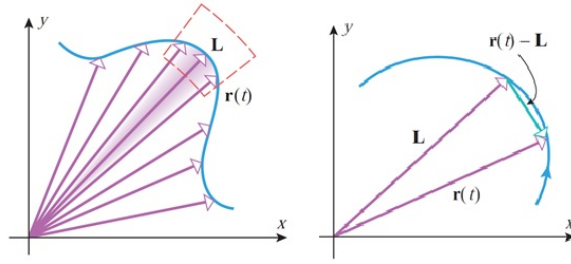


Figure 22.8: Left:  $\mathbf{r}(t)$  approaches  $\mathbf{L}$  in length and direction if  $\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{L}$ . Right:  $|\mathbf{r}(t) - \mathbf{L}|$  is the distance between terminal points for vectors  $\mathbf{r}(t)$  and  $\mathbf{L}$  when positioned with the same initial points.

**Theorem 22.11.** If  $\mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ , then

$$\begin{aligned} \lim_{t \rightarrow a} \mathbf{r}(t) &= \left( \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \right) \\ &= \lim_{t \rightarrow a} x(t)\hat{i} + \lim_{t \rightarrow a} y(t)\hat{j} + \lim_{t \rightarrow a} z(t)\hat{k} \end{aligned}$$

provided the limits of the component functions exist. Conversely, the limits of the component functions exist provided  $\mathbf{r}(t)$  approaches a limiting vector as  $t$  approaches  $a$ .

**Example 22.9.** Let  $\mathbf{r}(t) = t^2\hat{i} + e^t\hat{j} - (2\cos\pi t)\hat{k}$ . Then

$$\lim_{t \rightarrow 0} \mathbf{r}(t) = \left( \lim_{t \rightarrow 0} t^2 \right) + \left( \lim_{t \rightarrow 0} e^t \right) \hat{j} - \left( \lim_{t \rightarrow 0} 2\cos\pi t \right) \hat{k} = \hat{j} - 2\hat{k}.$$

**Definition 22.12** (Derivative of vector functions). If  $\mathbf{r}(t)$  is a vector-valued function, we define the derivative of  $\mathbf{r}$  with respect to  $t$  to be the vector-valued function  $\mathbf{r}'$  given by

$$\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

The domain of  $\mathbf{r}'$  consists of all values of  $t$  in the domain of  $\mathbf{r}(t)$  for which the limit exists.

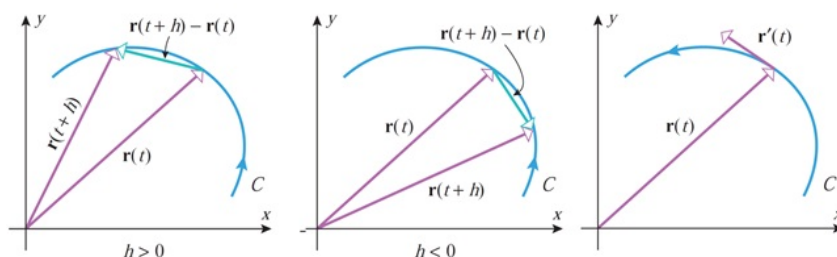


Figure 22.9: Idea of the derivative of a vector valued function.

**Theorem 22.13.** If  $\mathbf{r}(t)$  is a vector-valued function, then  $\mathbf{r}(t)$  is differentiable at  $t$  if and only if each of its component functions is differentiable at  $t$ , in which case the component functions of  $\mathbf{r}'(t)$  are the derivatives of the corresponding component functions of  $\mathbf{r}(t)$ .

**Example 22.10.** Let  $\mathbf{r}(t) = t^2\hat{i} + e^t\hat{j} - (2\cos\pi t)\hat{k}$ . Then

$$\mathbf{r}' = \frac{d}{dt}(t^2)\hat{i} + \frac{d}{dt}(e^t)\hat{j} - \frac{d}{dt}(2\cos\pi t)\hat{k} = 2t\hat{i} + e^t\hat{j} + (2\pi\sin\pi t)\hat{k}.$$

**Definition 22.14** (Tangent vector).

Let  $P$  be a point on the graph of a vector-valued function  $\mathbf{r}(t)$ , and let  $\mathbf{r}(t_0)$  be the radius vector from the origin to  $P$ . If  $\mathbf{r}'(t_0)$  exists and  $\mathbf{r}'(t_0) \neq 0$ , then we call  $\mathbf{r}'(t_0)$  a tangent vector to the graph of  $\mathbf{r}(t)$  at  $t_0$ , and we call the line through  $P$  that is parallel to the tangent vector the tangent line to the graph of  $\mathbf{r}(t)$  at  $\mathbf{r}(t_0)$ .

**Example 22.11.** Find parametric equations of the tangent line to the circular helix

$$x = \cos t, \quad y = \sin t, \quad z = t$$

where  $t = t_0$ , and find parametric equations for the tangent line at  $t = \pi$ .

*Solution.* The vector equation of the helix is

$$\mathbf{r}(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}.$$

Therefore,

$$\mathbf{r}(t) = \cos t_0\hat{i} + \sin t_0\hat{j} + t_0\hat{k}.$$

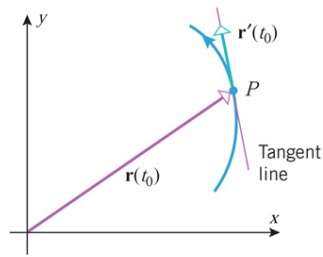
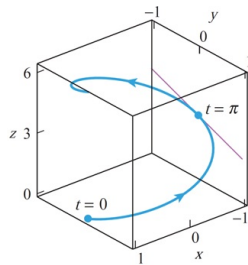


Figure 22.10: Tangent line and tangent vector.

and

$$\vec{v}_0 = \mathbf{r}'(t_0) = -\sin t_0 \hat{i} + \cos t_0 \hat{j} + \hat{k}.$$

Figure 22.11: Circular helix defined by  $x = \cos t$ ,  $y = \sin t$  and  $z = t$ .

It follows that the vector equation of the tangent line at  $t = t_0$  is

$$\begin{aligned} \mathbf{r}(t) &= (\cos t_0 \hat{i} + \sin t_0 \hat{j} + t_0 \hat{k}) + t [-\sin t_0 \hat{i} + \cos t_0 \hat{j} + \hat{k}] \\ &= (\cos t_0 - t \sin t_0) \hat{i} + (\sin t_0 + t \cos t_0) \hat{j} + (t_0 + t) \hat{k} \end{aligned}$$

Thus the parametric equations of the tangent line at  $t = t_0$  are

$$x = \cos t_0 - t \sin t_0, \quad y = \sin t_0 + t \cos t_0, \quad z = t_0 + t.$$

In particular, the tangent line at  $t = \pi$  has parametric equations

$$x = -1, \quad y = -t, \quad z = \pi + t.$$

□

**Definition 22.15.** Let  $\mathbf{r}_1(t)$  and  $\mathbf{r}_2(t)$  be two vector-valued differentiable functions, then

$$\begin{aligned} \frac{d}{dt} [\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)] &= \mathbf{r}_1(t) \cdot \frac{d}{dt} \mathbf{r}_2(t) + \frac{d}{dt} \mathbf{r}_1(t) \cdot \mathbf{r}_2(t) \\ \frac{d}{dt} [\mathbf{r}_1(t) \times \mathbf{r}_2(t)] &= \mathbf{r}_1(t) \times \frac{d}{dt} \mathbf{r}_2(t) + \frac{d}{dt} \mathbf{r}_1(t) \times \mathbf{r}_2(t) \end{aligned}$$

**Theorem 22.16.** If  $\mathbf{r}(t)$  is a differentiable vector-valued function and  $|\mathbf{r}(t)|$  is constant for all  $t$ , then

$$\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0,$$

that is,  $\mathbf{r}(t)$  and  $\mathbf{r}'(t)$  are orthogonal vectors for all  $t$ .

## 22.8 Integration

**Definition 22.17.**

Let  $\mathbf{r}(t)$  be an integrable vector-valued function, then

$$\int_a^b \mathbf{r}(t) dt = \left( \int_a^b x(t) dt \right) \hat{i} + \left( \int_a^b y(t) dt \right) \hat{j} + \left( \int_a^b z(t) dt \right) \hat{k}.$$

An antiderivative for a vector-valued function  $\mathbf{r}(t)$  is a vector-valued function  $\mathbf{R}(t)$  such that  $\mathbf{R}'(t) = \mathbf{r}(t)$ .

**Remark 22.18.** The vector form of fundamental theorem of Calculus is given by

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

**Example 22.12.** Evaluate the definite integral  $\int_0^2 (2t\hat{i} + 3t^2\hat{j}) dt$ .

*Solution.* Integrating the components yields

$$\int_0^2 (2t\hat{i} + 3t^2\hat{j}) dt = t^2 \Big|_0^2 \hat{i} + t^3 \Big|_0^2 \hat{j} = 4\hat{i} + 8\hat{j}.$$

□

**Example 22.13.** Find  $\mathbf{r}(t)$  given that  $\mathbf{r}'(t) = (3, 2t)$  and  $\mathbf{r}(1) = (2, 5)$ .

*Solution.* Integrating the  $\mathbf{r}'(t)$  to obtain  $\mathbf{r}(t)$  yields

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = \int (3, 2t) dt = (3t, t^2) + \mathbf{C}$$

where  $\mathbf{C}$  is a vector of integration. To find the value of  $\mathbf{C}$ , we substitute  $t = 1$  and use the given value of  $\mathbf{r}(1)$  to obtain  $\mathbf{r}(1) = (3, 1) + \mathbf{C} = (2, 5)$  so that  $\mathbf{C} = (-1, 4)$ . This,

$$\mathbf{r}(t) = (3t, t^2) + (-1, 4) = (3t - 1, t^2 + 4).$$

□